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HOMOGENEITY ON THREE-DIMENSIONAL CONTACT METRIC MANIFOLDS

BY

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ABSTRACT

We study ball-homogeneity, curvature homogeneity, natural reductivity, conformal flatness and φ-symmetry for three-dimensional contact metric manifolds. Several classification results are given.

1. Introduction

A Riemannian manifold such that the volume of all sufficiently small geodesic balls only depends on the radius is called a *ball-homogeneous* space [23]. Locally homogeneous spaces are trivial examples and, up to now, no other examples are known. This raises the question whether all ball-homogeneous spaces are locally homogeneous or not. Several affirmative answers have been obtained for special classes of manifolds but the general case remains open [12], [13], [14], [15]. Surprisingly, even in dimension three, a general answer is not known. This motivates the study of three-dimensional ball-homogeneous *contact metric manifolds* which we start in this paper. In particular, we consider the class of contact metric

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manifolds for which *the characteristic vector field is an eigenvector field of the Ricci tensor.* This is a condition which naturally appears in many problems and examples. Based on the results about three-dimensional homogeneous contact metric manifolds obtained in [28], we derive in Section 3 the classification of the above mentioned class of spaces and show that this class is formed by the spaces which are locally isometric to a unimodular Lie group equipped with a left-invariant contact metric structure. We also show that ball-homogeneity may be replaced by *curvature homogeneity*, that is, the eigenvalues of the Ricci tensor are constant.

In Section 3, we also make a further study of these unimodular Lie groups to determine which of them have a *cyclic-parallel Ricci tensor.* As a special case and using the Webster scalar curvature, we determine the class of *naturally reductive* ones.

Based on the derived formulas and results, we classify in Section 4 the *conformally flat* three-dimensional contact metric manifolds satisfying the already mentioned property for the characteristic vector field. This extends a result obtained by Tanno in [31].

Finally, in Section 5, we determine all three-dimensional *locally φ-symmetric spaces*, that is, the contact metric spaces such that the reflections with respect to the integral curves of the characteristic vector field are local isometries. This completes a result of [28].

2. Preliminaries

In this section we collect some basic facts about contact metric manifolds. All manifolds are assumed to be connected and smooth.

A $(2n+1)$-dimensional manifold M has an *almost contact structure* if it admits a vector field ξ (the *characteristic field*), a one-form η and a $(1,1)$-tensor field φ satisfying

$$\eta(\xi)=1, \quad \varphi^2=-\mathrm{id}+\eta\otimes\xi. \tag{2.1}$$

Then one can always find a Riemannian metric g which is compatible with the structure, that is, such that

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X and Y. (ξ,η,φ,g) is called an *almost contact metric structure* and (M,ξ,η,φ,g) an *almost contact metric manifold.* If additionally

$d\eta(X,Y) = g(X,\varphi Y)$ holds, then (M,ξ,η,φ,g) is called a *contact metric manifold.*

In what follows we denote by ∇ the Levi Civita connection and by R the corresponding Riemann curvature tensor given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all smooth vector fields X, Y. ρ denotes the Ricci tensor of type $(0,2)$ and Q the corresponding endomorphism field. We denote the scalar curvature by r and put $\sigma = \rho(\xi,\cdot)_{|\ker\eta}$.

On an almost contact metric manifold we have

$$\varphi\xi = 0, \quad \eta\circ\varphi = 0. \tag{2.3}$$

Moreover, if $\mathcal{L}$ denotes the Lie differentiation, we denote by ℓ and h the operators defined by

$$h = \frac{1}{2}\mathcal{L}_\xi\varphi, \quad \ell X = R(\xi,X)\xi, \quad \tau = \mathcal{L}_\xi g.$$

These $(1,1)$-type tensors h and ℓ are symmetric and satisfy

$$h\xi = 0, \quad \ell\xi = 0, \quad \mathrm{tr}\ h = 0, \quad \mathrm{tr}\, h\varphi = 0, \quad h\varphi = -\varphi h, \tag{2.4}$$

and, moreover, we have

$$\begin{aligned} &\nabla_X\xi = -\varphi X - \varphi h X,\\ &\nabla_\xi\varphi = 0,\\ &\mathrm{tr}\ \ell = \rho(\xi,\xi) = 2n - \mathrm{tr}\ h^2,\\ &\varphi\ell\varphi - \ell = 2(\varphi^2 + h^2),\\ &\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2. \end{aligned} \tag{2.5}$$

This last relation appears in [6] and for all the other formulas we refer to [3], where also more information may be found about contact geometry.

A contact metric space is said to be a *K-contact manifold* if ξ is a Killing vector field, that is, $\tau = 0$ or, equivalently, $h = 0$. Moreover, an almost contact metric structure (ξ,η,φ,g) is called a *Sasakian* structure if and only if

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.6}$$

Any Sasakian manifold is K-contact and the converse also holds when $n = 1$, that is, for three-dimensional spaces.

For a three-dimensional contact metric manifold, the *Webster scalar curvature* W is given by [17]

$$W = \frac{1}{8}(r - \rho(\xi,\xi) + 4) = \frac{1}{8}(r + 2 + \frac{1}{4}\|\tau\|^2). \tag{2.7}$$

The length $\|\tau\|$ is the *torsion invariant* introduced in [17].

Next, let $(M, \xi, \eta, \varphi, g)$ be a three-dimensional contact metric manifold and m a point of M. Then there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ in a neighborhood of m. Now, let $\mathcal{U}_1$ be the open subset of M where $h \neq 0$ and let $\mathcal{U}_2$ be the open subset of points $m \in M$ such that $h = 0$ in a neighborhood of m. $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open dense subset of M. On $\mathcal{U}_1$ we put $he = \lambda e$ and hence, from (2.4), we have $h\varphi e = -\lambda\varphi e$ where λ is a non-vanishing smooth function which we suppose to be positive. Then we have

LEMMA 2.1: *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi e = -a\varphi e, && \nabla_\xi \varphi e = ae,\\
&\nabla_e \xi = -(\lambda+1)\varphi e, && \nabla_{\varphi e}\xi = -(\lambda-1)e,\\
&\nabla_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}\varphi e, && \nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}e,\\
&\nabla_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}e + (\lambda+1)\xi, &&\\
&\nabla_{\varphi e} e = -\frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}\varphi e + (\lambda-1)\xi, &&
\end{aligned} \tag{2.8}$$

where a is a smooth function.

Proof: For any arbitrary three-dimensional contact metric space we have [32]

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$$

for all vector fields X, Y. Putting $X = e, Y = \xi$ and $X = \varphi e, Y = \xi$, we obtain

$$(\nabla_e \varphi)\xi = -(\lambda+1)e, \quad (\nabla_{\varphi e}\varphi)\xi = (\lambda-1)\varphi e.$$

This yields

$$\varphi\nabla_e\xi = (\lambda+1)e, \qquad \varphi\nabla_{\varphi e}\xi = -(\lambda-1)\varphi e$$

and hence, with (2.1), we get

$$\nabla_e\xi = -(\lambda+1)\varphi e, \quad \nabla_{\varphi e}\xi = -(\lambda-1)e.$$

Next, we recall that the curvature tensor of a three-dimensional Riemannian manifold satisfies

$$\begin{aligned} R(X,Y)Z =& g(X,Z)QY - g(Y,Z)QX - g(QY,Z)X + g(QX,Z)Y \\ & - \frac{r}{2}\{g(X,Z)Y - g(Y,Z)X\}. \end{aligned} \tag{2.9}$$

Putting $X = e, Y = \varphi e, Z = \xi$, we have

$$R(e,\varphi e)\xi = g(Qe,\xi)\varphi e - g(Q\varphi e,\xi)e = \sigma(e)\varphi e - \sigma(\varphi e)e. \tag{2.10}$$

On the other hand, we have [27, Proposition 3.1,(i)]

$$R(e,\varphi e)\xi = \varphi(\nabla_e h)\varphi e - \varphi(\nabla_{\varphi e} h)e. \tag{2.11}$$

Now, by a straightforward computation and using $\nabla_\xi \xi = 0$, we get

$$\begin{aligned} &\nabla_\xi e = -a\varphi e, && \nabla_\xi \varphi e = ae, \\ &\nabla_e e = b\varphi e, && \nabla_e \varphi e = -be + (\lambda+1)\xi, \\ &\nabla_{\varphi e}\varphi e = ce, && \nabla_{\varphi e} e = -c\varphi e + (\lambda - 1)\xi. \end{aligned} \tag{2.12}$$

From (2.11) we then obtain

$$R(e,\varphi e)\xi = \{-2\lambda c + e(\lambda)\}e + \{2\lambda b - (\varphi e)(\lambda)\}\varphi e$$

and, comparing this with (2.10), we have

$$2\lambda b = \sigma(e) + (\varphi e)(\lambda), \qquad 2\lambda c = \sigma(\varphi e) + e(\lambda).$$

Now, the required formulas (2.8) follow at once. ∎

Next, we derive a useful formula for $\nabla_\xi h$.

PROPOSITION 2.1: *On $\mathcal{U}_1$ we have*

$$\nabla_\xi h = 2ah\varphi + \xi(\lambda)s \tag{2.13}$$

where s is the $(1,1)$-type tensor defined by $s\xi = 0, se = e, s\varphi e = -\varphi e$.

Proof: Using (2.4), (2.5) and (2.8), we get

$$\begin{aligned} &(\nabla_\xi h)\xi = 0 = (2ah\varphi + \xi(\lambda)s)\xi, \\ &(\nabla_e h)e = -2a\lambda\varphi e + \xi(\lambda)e = (2ah\varphi + \xi(\lambda)s)e, \\ &(\nabla_\xi h)\varphi e = -2ahe - \xi(\lambda)\varphi e = (2ah\varphi + \xi(\lambda)s)\varphi e. \end{aligned}$$ ∎

Note that (2.13) holds trivially on $\mathcal{U}_2$ since $h = 0$.

Remark 2.1: $\nabla_\xi h = 0$ implies $a = \xi(\lambda) = 0$. In this case, the formulas (2.8) are given in Proposition 3.1 of [18].

Finally, we note that it follows from [27] that the Ricci operator Q on a three-dimensional contact metric space is given by

$$Q = \alpha I + \beta\eta \otimes \xi + \varphi\nabla_\xi h - \sigma(\varphi^2) \otimes \xi + \sigma(e)\eta \otimes e + \sigma(\varphi e)\eta \otimes \varphi e$$

where

$$\alpha = \frac{1}{2}r - 1 + \lambda^2, \quad \beta = -\frac{1}{2}r + 3 - 3\lambda^2. \tag{2.14}$$

Using (2.13), we get

$$Q = \alpha I + \beta\eta \otimes \xi + 2ah + \xi(\lambda)\varphi s - \sigma(\varphi^2) \otimes \xi + \sigma(e)\eta \otimes e + \sigma(\varphi e)\eta \otimes \varphi e, \tag{2.15}$$

and hence

$$\begin{aligned} Q\xi &= (\alpha + \beta)\xi + \sigma(e)e + \sigma(\varphi e)\varphi e, \\ Qe &= \sigma(e)\xi + (\alpha + 2a\lambda)e + \xi(\lambda)\varphi e, \\ Q\varphi e &= \sigma(\varphi e)\xi + \xi(\lambda)e + (\alpha - 2a\lambda)\varphi e. \end{aligned} \tag{2.16}$$

Note that $\sigma = 0$ *if and only if* ξ *is an eigenvector of* Q.

Moreover, since $\tau(X, Y) = 2g(\varphi X, hY)$ [27], we have

$$W = \frac{1}{8}(r + 2(1 + \lambda^2)), \qquad \lambda = \frac{\|\tau\|}{2\sqrt{2}}. \tag{2.17}$$

3. Ball- and curvature homogeneity, natural reductivity

In this section we consider different kinds of homogeneity and start with

Definition 3.1: A contact metric manifold $(M, \xi, \eta, \varphi, g)$ is said to be *homogeneous* if there exists a connected Lie group of isometries acting transitively on M and leaving η invariant. It is said to be *locally homogeneous* if the pseudogroup of local isometries acts transitively on M and leaves η invariant.

Note that a three-dimensional locally homogeneous contact metric manifold is locally isometric to a homogeneous one.

Homogeneous contact metric manifolds of dimension 3 have been studied in [28] by using Milnor's classification of Lie groups [24]. We recall the basic result.

THEOREM 3.1: *Let $(M, \xi, \eta, \varphi, g)$ be a three-dimensional, simply connected, homogeneous contact metric manifold. Then M is a Lie group and (η, g) is a left-invariant contact metric structure. More precisely, we have the following classification in terms of the Webster scalar curvature W and the torsion invariant $\|\tau\|$:*

(1) If G is unimodular, then it is one of the following Lie groups:
- *the Heisenberg group when $W = \|\tau\| = 0$;*
- *the 3-sphere group $\mathrm{SU}(2)$ when $4\sqrt{2}W > \|\tau\|$;*
- *the group $\tilde{E}(2)$, that is, the universal covering of the group of rigid motions of Euclidean 2-space, when $4\sqrt{2}W = \|\tau\| > 0$;*
- *the group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ when $-\|\tau\| \neq 4\sqrt{2}W < \|\tau\|$;*
- *the group $E(1,1)$ of rigid motions of Minkowski 2-space when $4\sqrt{2}W = -\|\tau\| < 0$. Moreover, in all these cases we have $\sigma = 0$.*

(2) If G is non-unimodular, its Lie algebra is given by

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = \gamma e_2, \quad [e_2, \xi] = 0,$$

where $\alpha \neq 0, e_1, e_2 = \varphi e_1 \in \ker \eta$ and $4\sqrt{2}W < \|\tau\|$. Moreover, if $\gamma = 0$ (or equivalently, $\sigma = 0$), then the structure is Sasakian and $W = -\alpha^2/4$.

Note that it is proved in [21] that the non-unimodular Lie group satisfying $\sigma = 0$ is isometric to a unimodular group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. We also refer to [28] for the explicit construction of homogeneous contact metric structures on these Lie groups.

Now, we turn to the consideration of *ball-homogeneous* contact metric spaces of dimension 3 as mentioned in the Introduction. We note that ball-homogeneity implies the constancy of an infinite number of scalar curvature invariants. See for example [13]. In particular, r and $\|\rho\|^2$ must be constant. Now, we prove

THEOREM 3.2: *A three-dimensional contact metric manifold $(M, \xi, \eta, \varphi, g)$ is locally isometric to a unimodular Lie group equipped with a left-invariant contact metric structure if and only if it is ball-homgeneous and $\sigma = 0$.*

Proof: The "only if" part follows from the local homogeneity of the manifold and from Theorem 3.1.

To prove the "if" part, we first consider the case $M = \mathcal{U}_2$, that is, (ξ, η, φ, g) is a Sasakian structure. Since r is constant, it then follows that M is locally φ-symmetric [34], and hence locally homogeneous [30]. The result then follows again from Theorem 3.1.

Next, assume that $\mathcal{U}_1$ is not empty and let $\{\xi, e, \varphi e\}$ be a basis as in Section 2. Since $\sigma = 0$, (2.16) yields

$$\begin{aligned} Q\xi &= 2(1-\lambda^2)\xi, \\ Qe &= (\alpha + 2a\lambda)e + \xi(\lambda)\varphi e, \\ Q\varphi e &= \xi(\lambda)e + (\alpha - 2a\lambda)\varphi e. \end{aligned} \tag{3.1}$$

Further, we have the well-known formula

$$\frac{1}{2}X(r) = \sum_{i=1}^{n} g((\nabla_{e_i}Q)e_i, X) \tag{3.2}$$

for any vector field X and any n-dimensional Riemannian manifold. Here, $\{e_i, i = 1, \ldots, n\}$ is an arbitrary orthonormal basis. In our case, from (3.1), (2.8), (2.5), we get

$$\begin{aligned} (\nabla_\xi Q)\xi =& -4\lambda\xi(\lambda)\xi, \\ (\nabla_e Q)e =& \{e(\alpha + 2a\lambda) - \frac{1}{\lambda}(\varphi e)(\lambda)\xi(\lambda)\}e + \{e(\xi(\lambda)) + 2a(\varphi e)(\lambda)\}\varphi e \\ &+ (\lambda+1)\xi(\lambda)\xi, \\ (\nabla_{\varphi e} Q)\varphi e =& \{(\varphi e)(\alpha - 2a\lambda) - \frac{1}{\lambda}e(\lambda)\xi(\lambda)\}\varphi e + \{(\varphi e)(\xi(\lambda)) - 2ae(\lambda)\}e \\ &+ (\lambda - 1)\xi(\lambda)\xi. \end{aligned}$$

From these relations and (3.2) we then get

$$\frac{1}{2}\xi(r) = -2\lambda\xi(\lambda), \tag{3.3}$$

$$\frac{1}{2}e(r) = e(\alpha + 2a\lambda) - \frac{1}{\lambda}\xi(\lambda)(\varphi e)(\lambda) + (\varphi e)\xi(\lambda) - 2ae(\lambda), \tag{3.4}$$

$$\frac{1}{2}(\varphi e)(r) = (\varphi e)(\alpha - 2a\lambda) - \frac{1}{\lambda}\xi(\lambda)e(\lambda) + e\xi(\lambda) + 2a(\varphi e)(\lambda). \tag{3.5}$$

Since r is constant and $\lambda \neq 0$, (3.4) and (3.5) yield

$$e(\lambda) + e(a) = 0, \quad (\varphi e)(\lambda) - (\varphi e)(a) = 0. \tag{3.6}$$

Next, we compute $\|\rho\|^2$ from (2.16) taking into account that $\xi(\lambda) = 0$ follows from (3.3). We get

$$\|\rho\|^2 = 4(1-\lambda^2)^2 + 2(\frac{r}{2} - 1 + \lambda^2)^2 + 8a^2\lambda^2.$$

Since r and $||\rho||^2$ are constant, we then get

$$6\lambda^4 + 2(r-6)\lambda^2 + 8a^2\lambda^2 = \text{ const.} \tag{3.7}$$

Taking the derivative of (3.7) with respect to e yields

$$e(\lambda)\{12\lambda^2 + 2(r-6) - 8a\lambda + 8a^2\} = 0$$

and this implies $e(\lambda) = 0$. Indeed, if $e(\lambda) \neq 0$ in a neighborhood, we have

$$12\lambda^2 + 2(r-6) - 8a\lambda + 8a^2 = 0$$

and a new differentiation in the direction of e gives $4\lambda - 3a = 0$. Deriving once again and using (3.6) then gives $e(\lambda) = 0$, which contradicts the hypothesis. Using (3.6), we then get $e(\lambda) = e(a) = 0$. A similar reasoning also gives $(\varphi e)(\lambda) = (\varphi e)(a) = 0$. Moreover, from (2.8) we get $2\xi = [e, \varphi e]$, and hence we obtain $\xi(a) = 0$. All this implies that λ and a are locally constant on $\mathcal{U}_1$. Since λ is continuous, we get from this that $M = \mathcal{U}_1$, and hence λ and a are globally constant.

So, from (2.8), we obtain

$$[e, \varphi e] = c_1\xi, \quad [\varphi e, \xi] = c_2 e, \quad [\xi, e] = c_3\varphi e,$$

where $c_1 = 2$, $c_2 = 1 - \lambda - a$, $c_3 = \lambda + 1 - a$ are constants. From this we may conclude that $(M, \xi, \eta, \varphi, g)$ is locally isometric to a unimodular Lie group with a left-invariant contact metric structure. See for example [33, p. 10] and [24] or Theorem 3.1. ∎

Remark 3.1: We note that the proof of Theorem 3.2 shows that if $\sigma = 0$ and $r =$ const., then $\{\xi, e, \varphi e\}$ is a basis of eigenvectors of Q on $\mathcal{U}_1$.

As already mentioned in the Introduction, a three-dimensional manifold is *curvature homogeneous* if the eigenvalues of the Ricci operator Q are constant. Then r and $||\rho||^2$ are constant. The proof of Theorem 3.2 then shows the validity of the following result.

THEOREM 3.3: *Let $(M, \xi, \eta, \varphi, g)$ be a three-dimensional contact metric manifold. Then the following statements are equivalent:*

(i) *$(M, \xi, \eta, \varphi, g)$ is locally isometric to a unimodular Lie group equipped with a left-invariant contact metric structure;*

(ii) *$(M, \xi, \eta, \varphi, g)$ is ball-homogeneous and $\sigma = 0$;*

(iii) $(M, \xi, \eta, \varphi, g)$ *is curvature homogeneous and* $\sigma = 0$;
(iv) $\sigma = 0$ *and* r *and* $\|\rho\|^2$ *are constant on* $(M, \xi, \eta, \varphi, g)$.

Note that the case $\sigma = 0$ and λ constant, that is, ξ is an eigenvector of Q with constant eigenvalue, has been treated in [20].

Now, we want to determine the naturally reductive three-dimensional contact metric manifolds. Since natural reductivity implies that ρ is a Killing tensor, that is, ρ is cyclic-parallel [22], we first consider this last condition and prove

THEOREM 3.4: *Let* $(M, \xi, \eta, \varphi, g)$ *be a three-dimensional contact metric manifold. Then* $\sigma = 0$ *and* ρ *is cyclic-parallel if and only if the manifold is locally isometric to a unimodular Lie group* G *equipped with a left-invariant metric structure and satisfying one of the following conditions:*

(i) $\tau = 0$ *(that is, the structure is Sasakian). Then* G *is the Heisenberg group* H_3 *if* $W = 0$, $\mathrm{SU}(2)$ *if* $W > 0$ *or* $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ *if* $W < 0$;
(ii) $\tau \neq 0$ *and* $W = 1 + \|\tau\|/4\sqrt{2}$. *In this case* $G = \mathrm{SU}(2)$;
(iii) $\tau \neq 0$ *and* $W = 1 - \|\tau\|/4\sqrt{2}$. *Then* G *is* $\mathrm{SU}(2)$ *if* $\|\tau\| < 2\sqrt{2}$, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ *if* $\|\tau\| > 2\sqrt{2}$ *or* $\tilde{E}(2)$ *if* $\|\tau\| = 2\sqrt{2}$. *(In this last case,* g *is a flat metric.)*

Proof: First, assume $\sigma = 0$ and ρ cyclic-parallel. If $\tau = 0$, the structure is Sasakian. Moreover, since ρ is cyclic-parallel, r is constant. Hence, $(M, \xi, \eta, \varphi, g)$ is locally homogeneous [30], [34]. So, Theorem 3.1 implies that M is locally isometric to a Lie group equipped with a left-invariant contact metric structure. More precisely, we have $G =$ Heisenberg group H_3 if $W = 0$, $G = \mathrm{SU}(2)$ if $W > 0$ and $G = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ if $W < 0$.

Next, let $\mathcal{U}_1$ be non-empty. From (2.15) and Remark 3.1 we then get $\xi(\lambda) = 0$ and $Q = \alpha I + \beta\eta \otimes \xi + 2ah$ on $\mathcal{U}_1$ and, using (2.8), we obtain

$$
\begin{aligned}
&(\nabla_\xi \rho))(\xi, e) = 0, \quad (\nabla_\xi \rho)(\xi, \varphi e) = 0,\\
&(\nabla_e \rho)(\xi, \xi) = -4\lambda e(\lambda), \quad (\nabla_{\varphi e} \rho)(\xi, \xi) = -4\lambda(\varphi e)(\lambda),\\
&(\nabla_e \rho)(e, e) = e(\lambda^2 + 2a\lambda), \quad (\nabla_{\varphi e} \rho)(\varphi e, \varphi e) = (\varphi e)(\lambda^2 - 2a\lambda),\\
&(\nabla_\xi \rho)(e, e) = 2\lambda\xi(a), \quad (\nabla_e \varphi)(e, \xi) = 0.
\end{aligned}
$$

Since ρ is cyclic-parallel, these relations yield

$$
\begin{aligned}
&e(\lambda) = 0, \quad (\varphi e)(\lambda) = 0,\\
&e(\lambda^2 + 2a\lambda) = 0, \quad (\varphi e)(\lambda^2 - 2a\lambda) = 0,\\
&\xi(a) = 0.
\end{aligned}
$$

Hence, λ and a are locally constant on $\mathcal{U}_1$ and, as in the proof of Theorem 3.2, we get that $(M, \xi, \eta, \varphi, g)$ is locally isometric to a unimodular Lie group equipped with a left-invariant contact metric structure.

Now, we express that ρ is cyclic-parallel by using the Webster scalar curvature. Therefore, we note that formula (5.2) of [26] yields

$$r = 4(1-\lambda^2) + 2H,$$

where $H = g(R(e,\varphi e)e,\varphi e)$ denotes the φ-sectional curvature. But from (2.8) we get, when $\sigma = 0$ and $\lambda =$ const.,

$$R(e,\varphi e)e = (\lambda^2 - 1 - 2a)\varphi e,$$

and hence $H = \lambda^2 - 1 - 2a$, which implies

$$2a = -\frac{r}{2} + 1 - \lambda^2. \tag{3.8}$$

Furthermore, from (2.14), (2.15), (2.17) and (3.8) we then get

$$Q = 2(2W-1)I + [2 - 4W + 2(1-\lambda^2)]\eta \otimes \xi + 2(1-2W)h.$$

It is now easy to check that the only non-vanishing components of $\nabla\rho$ are given by

$$(\nabla_\xi \rho)(e,\varphi e) = -4\lambda(1-2W)^2,$$
$$(\nabla_e \rho)(\xi,\varphi e) = 2(\lambda+1)^2(2W-2+\lambda),$$
$$(\nabla_{\varphi e} \rho)(\xi, e) = 2(\lambda-1)^2(-2W+\lambda+2).$$

Hence, if ρ is cyclic-parallel, we have $2W = 2 \pm \lambda$. Then, from Theorem 3.1 we get the required results concerning the group G.

To finish the proof, we show the existence of these left-invariant contact metric structures on $H_3, \mathrm{SU}(2), \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $\tilde{E}(2)$ satisfying the conditions for W and $\|\tau\|$. To see this, we note that the Lie algebra for these groups has the form

$$[e_2,e_3] = \lambda_1 e_1, \quad [e_3,e_1] = \lambda_2 e_2, \quad [e_1,e_2] = \lambda_3 e_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants [24]. Let $\{\theta^1,\theta^2,\theta^3\}$ be the dual basis of $\{e_1,e_2,e_3\}$. Then $d\theta^1 = -\lambda_1\theta^2 \wedge \theta^3$ and hence, if $\lambda_1 \neq 0$, $\eta = \theta^1$ is a contact form. For $\lambda_1 = 2$, the Riemannian metric defined by $g(e_i,e_j) = \delta_{ij}$, $i,j = 1,2,3$, satisfies $d\eta = g(\cdot,\varphi\cdot)$ where φ is defined by $\varphi e_1 = 0, \varphi e_2 = e_3, \varphi e_3 = -e_2$. So, g is a compatible metric. Moreover, we have [28]

$$2W = \frac{\lambda_2 + \lambda_3}{2}, \qquad \|\tau\| = \sqrt{2}(\lambda_3 - \lambda_2).$$

Following Milnor's classification (see also Theorem 3.1), the cases $\lambda_2 = \lambda_3, \lambda_2 = \lambda_1 = 2$ and $\lambda_3 = \lambda_1 = 2$ correspond to (i), (ii) and (iii), respectively. This completes the proof. ∎

In [1], it has been proved that a locally homogeneous three-dimensional Riemannian manifold with cyclic-parallel Ricci tensor is locally isometric to a naturally reductive homogeneous space. Hence, it follows from Theorem 3.4 that a three-dimensional contact metric space with cyclic-parallel Ricci tensor and $\sigma = 0$ is locally isometric to a naturally reductive space. Using the Webster scalar curvature, we have the following more specific result.

THEOREM 3.5: *Let $(M, \xi, \eta, \varphi, g)$ be a three-dimensional, simply connected, complete contact metric manifold. Then we have*

(i) *if $\tau = 0$, then $(M, \xi, \eta, \varphi, g)$ is naturally reductive if and only if W is constant;*
(ii) *if $\tau \neq 0$, then $(M, \xi, \eta, \varphi, g)$ is naturally reductive if and only if $\sigma = 0$ and $W = 1 \pm \|\tau\|/4\sqrt{2} =$ constant.*

If $(M, \xi, \eta, \varphi, g)$ is not simply connected or complete, then "naturally reductive" has to be replaced by "locally isometric to a naturally reductive space".

Proof: First, let $\tau = 0$, that is, the structure is Sasakian. Then $8W = r + 2$. Hence, W is constant if and only if the scalar curvature is constant. If r is constant, then the manifold is locally φ-symmetric and hence locally isometric to a naturally reductive space [8]. The converse is trivial.

Next, let $\tau \neq 0$. If $(M, \xi, \eta, \varphi, g)$ is naturally reductive, then ρ is cyclic-parallel. Moreover, it follows from [2] that $\sigma = 0$. The value of W then follows from Theorem 3.4. Conversely, let $\sigma = 0$ and $W = 1 \pm \|\tau\|/4\sqrt{2} =$ constant. (2.17) implies that λ is constant, and hence, also r. Furthermore, (3.8) implies that a is constant. Now, as in the proof of Theorem 3.2, it follows that the manifold is a homogeneous contact metric space and then the result follows from Theorem 3.4. ∎

4. Conformally flat contact metric manifolds

In [31], Tanno proved that a conformally flat K-contact space has constant sectional curvature +1. See [25] for the Sasakian case and dimension ≥ 5. On the other hand, it is shown in [4] that there exist three-dimensional conformally flat contact metric spaces which are not real space forms. Now we show that this cannot occur when $\sigma = 0$.

THEOREM 4.1: *A three-dimensional conformally flat contact metric space with $\sigma = 0$ has constant sectional curvature* 0 *or* 1.

Proof: First, let $M = \mathcal{U}_2$. Then the result follows from Tanno's work [31].

Next, suppose $\mathcal{U}_1$ is not empty. In what follows we shall show that λ and a are constant. It then follows again that the manifold is locally homogeneous. Since it is conformally flat, it must be locally symmetric [29]. Then the result follows from [7].

To prove that λ and a are constant, we take the usual basis $\{\xi, e, \varphi e\}$ and put $e_1 = \xi$, $e_2 = e$, $e_3 = \varphi e$. Using the notational convention $\rho_{ij} = \rho(e_i, e_j), \nabla_i\rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$, a straightforward computation then yields

$$
\begin{aligned}
\nabla_1\rho_{11} &= -4\lambda\xi(\lambda),\\
\nabla_1\rho_{12} &= \nabla_1\rho_{13} = 0,\\
\nabla_1\rho_{22} &= \xi(\rho_{22}) + 2a\rho_{23},\\
\nabla_1\rho_{33} &= \xi(\rho_{33}) - 2a\rho_{23},\\
\nabla_2\rho_{11} &= e(\rho_{11}),\\
\nabla_2\rho_{12} &= (1+\lambda)\rho_{23},\\
\nabla_2\rho_{13} &= (1+\lambda)(\rho_{33} - \rho_{11}),\\
\nabla_2\rho_{33} &= e(\rho_{33}) + \frac{1}{\lambda}(\varphi e)(\lambda)\rho_{23},\\
\nabla_2\rho_{23} &= e(\rho_{23}) + 2a(\varphi e)(\lambda),\\
\nabla_3\rho_{11} &= (\varphi e)(\rho_{11}),\\
\nabla_3\rho_{12} &= (1-\lambda)(\rho_{11} + \rho_{22}),\\
\nabla_3\rho_{13} &= (\lambda - 1)\rho_{23},\\
\nabla_3\rho_{22} &= (\varphi e)(\rho_{22}) + \frac{1}{\lambda}e(\lambda)\rho_{23},\\
\nabla_3\rho_{23} &= (\varphi e)(\rho_{23}) - 2ae(\lambda).
\end{aligned}
\tag{4.1}
$$

We shall also use extensively the components ρ_{ij} given by (2.16).

Now, since (M, g) is conformally flat, we have

$$
\nabla_k\rho_{ij} - \nabla_j\rho_{ik} = \frac{1}{4}(\delta_{ij}\nabla_k r - \delta_{ik}\nabla_i r), \qquad i, j, k = 1, 2, 3. \tag{4.2}
$$

Hence, from (4.1) and (4.2) we get

$$
\begin{aligned}
\frac{1}{4}\xi(r) &= \xi(\rho_{22}) + (2a - \lambda - 1)\rho_{23},\\
\frac{1}{4}\xi(r) &= \xi(\rho_{33}) + (1 - \lambda - 2a)\rho_{23},\\
2\rho_{11} &= (1-\lambda)\rho_{22} + (1+\lambda)\rho_{33}.
\end{aligned}
\tag{4.3}
$$

From these relations, we obtain

$$\xi(\rho_{22}-\rho_{33}) = 2(1-2a)\rho_{23},$$
$$\xi(\rho_{22}+\rho_{33}) = 2\lambda\rho_{23} + \frac{1}{2}\xi(r). \tag{4.4}$$

Hence, we have

$$\xi(r) = \xi(\rho_{11}+\rho_{22}+\rho_{33}) = \xi(\rho_{11}) + 2\lambda\rho_{23} + \frac{1}{2}\xi(r).$$

Using (2.16), we then get $\xi(r) = -4\lambda\xi(\lambda) = \xi(\rho_{11})$. Hence, $\xi(\rho_{22}+\rho_{33}) = 0$. Now, we derive (4.3) with respect to ξ and use (4.4) to get $\xi(\rho_{11}) = -\lambda\xi(\lambda)$. This yields $\xi(\lambda) = 0$, and hence we have $\xi(r) = \xi(\rho_{11}) = \xi(\rho_{22}) = \xi(\rho_{33}) = 0$. Since $0 = \xi(\rho_{22}) = \xi(\alpha + 2a\lambda) = 2\lambda\xi(a)$, we also get $\xi(a) = 0$.

Further, from $\nabla_2\rho_{11} - \nabla_1\rho_{12} = \frac{1}{4}\nabla_2 r$ we get

$$e(\rho_{11}) = \frac{1}{4}e(r) \tag{4.5}$$

and from $\nabla_2\rho_{23} - \nabla_3\rho_{23} = \frac{1}{4}\nabla_2 r$ we obtain, taking into account that $\xi(\lambda) = \rho_{23} = 0$,

$$\begin{aligned}\frac{1}{4}e(r) &= e(\rho_{33}) + 2ae(\lambda)\\ &= \frac{1}{2}e(r) + 2\lambda e(\lambda) - 2\lambda e(a),\end{aligned}$$

and so

$$\frac{1}{4}e(r) = 2\lambda\{e(a) - e(\lambda)\}.$$

But with (4.5) we have $\frac{1}{4}e(r) = e(\rho_{11}) = -4\lambda e(\lambda)$, and hence $e(\lambda) + e(a) = 0$.

Next, from (4.3) we obtain

$$r = \frac{1}{2}(3-\lambda)\rho_{22} + \frac{1}{2}(3+\lambda)\rho_{33}.$$

Derive this with respect to e. This gives

$$e(r) = -4\lambda(2a-\lambda)e(\lambda)$$

which, together with $\frac{1}{4}e(r) = -4\lambda e(\lambda)$, yields

$$(2a - \lambda - 4)e(\lambda) = 0.$$

From this it follows that $e(\lambda) = 0$. Indeed, for $e(\lambda) \neq 0$ in a neighborhood, we get $\lambda = 2a - 4$ and differentiation then gives $e(\lambda) = 0$, which is a contradiction. So, we must have $e(\lambda) = e(a) = 0$.

In a similar way and using $\nabla_3\rho_{11} - \nabla_1\rho_{13} = \frac{1}{4}\nabla_3 r$ and $\nabla_3\rho_{22} - \nabla_2\rho_{23} = \frac{1}{4}\nabla_3 r$, we get $(\varphi e)(\lambda) = (\varphi e)(a) = 0$.

All this implies that λ and a are constant and this completes the proof. ∎

It is well-known that the Ricci tensor of a three-dimensional manifold is a Codazzi tensor if and only if it is conformally flat and has constant scalar curvature (see (4.2)). Hence, from Theorem 4.1, we obtain at once

THEOREM 4.2: *A three-dimensional contact metric space with $\sigma = 0$, and such that its Ricci tensor is a Codazzi tensor, has constant sectional curvature 0 or 1.*

5. Three-dimensional locally φ-symmetric spaces

For a Sasakian or a K-contact manifold, local symmetry implies that the manifold has constant curvature 1 [25], [31]. For this reason, *locally φ-symmetric spaces* have been introduced in [30] and, in [8], it has been shown that they may be defined as Sasakian manifolds such that the local reflections with respect to the integral curves of the characteristic vector field are isometries. Many examples are known and their classification has been treated in [19]. Using this property as definition, locally φ-symmetric spaces have also been introduced for contact metric spaces in [11]. In this section we shall derive two new characterizations for the three-dimensional contact metric case.

First, we have

THEOREM 5.1: *A three-dimensional contact metric manifold $(M, \xi, \eta, \varphi, g)$ is locally φ-symmetric if and only if it is locally homogeneous and $\sigma = 0$.*

Proof: First, suppose that the manifold is Sasakian. Then the result is well-known [30], [34].

Next, suppose that the contact metric structure on M is not Sasakian and let M be locally φ-symmetric. Since the local reflections are isometries, we have $\rho(\xi, u) = 0$ for any $u \in \ker\eta$, and hence $\sigma = 0$. Moreover, we must also have

$$\nabla_u r = 0,$$
$$(\nabla_u\rho)(u, u) = 0,$$
$$(\nabla_u\rho)(\xi, \xi) = 0.$$

Now, we show that this implies that λ and a, for the usual basis $\{\xi, e, \varphi e\}$ on $\mathcal{U}_1$, are constant. This implies the local homogeneity.

First, we have $0 = (\nabla_e \rho)(\xi,\xi) = -4\lambda e(\lambda)$ and so $e(\lambda) = 0$. Similarly, $(\nabla_{\varphi e}\rho)(\xi,\xi) = 0$ implies $(\varphi e)(\lambda) = 0$. This, (2.8) and $\sigma = 0$ give

$$\nabla_e \varphi e = (\lambda+1)\xi, \quad \nabla_{\varphi e} e = (\lambda - 1)\xi.$$

Hence, $[e, \varphi e] = 2\xi$ and so $\xi(\lambda) = 0$. Moreover, we also have $0 = (\nabla_e \rho)(e,e) = 2\lambda e(a)$, $0 = (\nabla_{\varphi e}\rho)(\varphi e, \varphi e) = -2\lambda(\varphi e)(a)$ and this yields $e(a) = (\varphi e)(a) = 0$, and hence also $\xi(a) = 0$. This implies that λ and a are constant on M.

Next, let (M,ξ,η,φ,g) be locally homogeneous. Then it is analytic and, since the integral curves of ξ are geodesics, [16] implies that the manifold is locally φ-symmetric if and only if

$$\text{(5.1)}\qquad \begin{aligned} &\text{(i)} && g((\nabla^{2k}_{u\dots u}R)(u,v)u,\xi) = 0,\\ &\text{(ii)} && g((\nabla^{2k+1}_{u\dots u}R)(u,v)u,w) = 0,\\ &\text{(iii)} && g((\nabla^{2k+1}_{u\dots u}R)(u,\xi)u,\xi) = 0, \end{aligned}$$

for all $u,v,w \in \ker\eta$ and all $k \in \mathbb{N}$. Since $\dim M = 3$, (2.9) holds and since $\sigma = 0$, this implies that (i) is satisfied for $k = 0$. Furthermore, since r is constant, we get from (2.9):

$$\text{(5.2)}\qquad \begin{aligned} (\nabla^{\ell}_{V\dots V}R)(X,Y)Z =& g(X,Z)(\nabla^{\ell}_{V\dots V}Q)Y - g(Y,Z)(\nabla^{\ell}_{V\dots V}Q)X\\ &- g((\nabla^{\ell}_{V\dots V}Q)Y,Z)X + g((\nabla^{\ell}_{V\dots V}Q)X,Z)Y \end{aligned}$$

for all $\ell \in \mathbb{N}_0$. In particular, we have

$$\begin{aligned} g((\nabla^{2k}_{u\dots u}R)(u,v)u,\xi) =& g_{uu}(\nabla^{2k}_{u\dots u}\rho)(v,\xi) - g_{uv}(\nabla^{2k}_{u\dots u}\rho)(u,\xi),\\ g((\nabla^{2k+1}_{u\dots u}R)(u,v)u,w) =& g_{uu}(\nabla^{2k+1}_{u\dots u}\rho)(v,w) - g_{uv}(\nabla^{2k+1}_{u\dots u}\rho)(u,u)\\ &- g_{uw}(\nabla^{2k+1}_{u\dots u}\rho)(u,v) + g_{vw}(\nabla^{2k+1}_{u\dots u}\rho)(u,u),\\ g((\nabla^{2k+1}_{u\dots u}R)(u,\xi)u,\xi) =& g_{uu}(\nabla^{2k+1}_{u\dots u}\rho)(\xi,\xi) + g_{\xi\xi}(\nabla^{2k+1}_{u\dots u}\rho)(u,u). \end{aligned}$$

Hence, it follows that (i), (ii) and (iii) are satisfied if

$$\begin{aligned} &\text{(i)}' && (\nabla^{2k}_{u\dots u}\rho)(v,\xi) = 0,\\ &\text{(ii)}' && (\nabla^{2\ell+1}_{u\dots u}\rho)(v,w) = (\nabla^{2\ell+1}_{u\dots u}\rho)(\xi,\xi) = 0, \end{aligned}$$

for all $u,v,w \in \ker\eta$ and all $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$.

To prove that (i)′ and (ii)′ hold, we have to consider the covariant derivatives of ρ of type $\nabla^{\ell}_{X_1\dots X_\ell}\rho$ where $X_i = \xi$ or $X_i \in \ker\eta$ for all $\ell \in \mathbb{N}_0$. To do this, we choose a basis $e_1 = \xi$, $e_2 = e$, $e_3 = \varphi e$ as before and denote by $\theta^1, \theta^2, \theta^3$ the dual basis. We introduce the following notations:

$$\alpha_1 = \theta^1 \otimes \theta^1, \quad \alpha_2 = \theta^2 \otimes \theta^2, \quad \alpha_3 = \theta^3 \otimes \theta^3, \quad \alpha_4 = \theta^2 \otimes \theta^3 + \theta^3 \otimes \theta^2,$$
$$\beta_1 = \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1, \quad \beta_2 = \theta^1 \otimes \theta^3 + \theta^3 \otimes \theta^1,$$
$$A = \text{ span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \quad B = \text{ span}\{\beta_1, \beta_2\}.$$

Note that since $\sigma = 0$, $\rho \in A$. Furthermore, the one-forms $\nabla_{e_i}\theta^j$ are the dual forms of $\nabla_{e_i}e_j$ for $i, j = 1, 2, 3$. They may be computed by using (2.8). Since $\sigma = 0$ and $\lambda =$ const., we obtain

$$\begin{aligned}
&\nabla_{e_1}\theta^1 = \nabla_{e_2}\theta^2 = \nabla_{e_3}\theta^3 = 0,\\
&\nabla_{e_1}\theta^2 = -a\theta^3, \quad \nabla_{e_1}\theta^3 = a\theta^2,\\
&\nabla_{e_2}\theta^1 = -(\lambda+1)\theta^3, \quad \nabla_{e_2}\theta^3 = (\lambda+1)\theta^1,\\
&\nabla_{e_3}\theta^1 = -(\lambda-1)\theta^2, \quad \nabla_{e_3}\theta^2 = (\lambda-1)\theta^1.
\end{aligned}$$

Then, a straightforward computation yields

$$\begin{aligned}
\nabla_\xi \alpha_i &\in A, & \nabla_\xi \beta_j &\in B,\\
\nabla_e \alpha_i &\in B, & \nabla_e \beta_j &\in A,\\
\nabla_{\varphi e} \alpha_i &\in B, & \nabla_{\varphi e} \beta_j &\in A,
\end{aligned}$$

for $i = 1, 2, 3, 4$ and $j = 1, 2$. Equivalently, we have

$$\nabla_\xi(A) \subset A, \quad \nabla_\xi(B) \subset B, \quad \nabla_u(A) \subset B, \quad \nabla_u(B) \subset A,$$

for all $u \in \ker\eta$. To proceed, we prove

LEMMA 5.1: *Let k be the number of $X_i \in \ker\eta$ appearing in $\nabla^\ell_{X_1\ldots X_\ell}\rho$. Then $\nabla^\ell_{X_1\ldots X_\ell}\rho \in A$ (respectively B) if and only if k is even (respectively odd).*

Proof: Since $\rho \in A$, the result holds for $\ell = 0$. Now, we proceed by induction and suppose that the result holds for ℓ. Further, we have

$$\nabla^{\ell+1}_{XX_1\ldots X_\ell}\rho = \nabla_X(\nabla^\ell_{X_1\ldots X_\ell}\rho) - \nabla^\ell_{\nabla_X X_1 X_2\ldots X_\ell}\rho \cdots - \nabla^\ell_{X_1\ldots\nabla_X X_\ell}\rho.$$

Now, we treat the cases $X = \xi$ and $X \in \ker\eta$ separately.

(a) Suppose $X = \xi$ and k even. The case k odd may be treated similarly. If $X_i = \xi$, then $\nabla_\xi X_i = 0$ while for $X_i \in \ker\eta$ we have $\nabla_\xi X_i \in \ker\eta$. Therefore $\nabla^\ell_{X_1\ldots X_{i-1}\nabla_X X_i X_{i+1}\ldots X_\ell}\rho$ vanishes or belongs to A. Further, since $\nabla^\ell_{X_1\ldots X_\ell}\rho \in A$ and $\nabla_\xi(A) \subset A$, we obtain $\nabla_\xi(\nabla^\ell_{X_1\ldots X_\ell}\rho) \in A$ and hence $\nabla^{\ell+1}_{\xi X_1\ldots X_\ell}\rho \in A$.

(b) Now, let $X \in \ker \eta$ and let k be even. For $X_i = \xi$ we have $\nabla_X X_i \in \ker \eta$ while if $X_i \in \ker \eta$, $\nabla_X X_i$ is proportional to ξ. The rest follows now by proceeding as in (a). ∎

Using this lemma, we are now able to complete the proof of the theorem. Indeed, let $u \in \ker \eta$. Then $\nabla^{2k}_{u\ldots u}\rho \in A$. Since $\alpha_i(v,\xi) = 0$ for $v \in \ker \eta$, (i)′ follows at once. Further, $\nabla^{2k+1}_{u\ldots u}\rho \in B$ and, since $\beta_j(v,w) = \beta_j(\xi,\xi) = 0$ for all $v, w \in \ker \eta$, we get (ii)′. ∎

To obtain a second characterization, we consider contact metric *manifolds whose characteristic vector field* ξ *belongs to the* (k,μ)*-nullity distribution* for some real numbers k and μ. This means that the curvature tensor R satisfies

$$R(X,Y)\xi = k\{\eta(X)Y - \eta(Y)X\} + \mu\{\eta(X)hY - \eta(Y)hX\} \tag{5.3}$$

for all vector fields X, Y. Typical examples are unit tangent sphere bundles of spaces of constant curvature and their D-homothetic transformed ones. We refer to [5], [9], [20] for more details about the geometry of these spaces.

Now, we prove

THEOREM 5.2: *Let* $M(\xi,\eta,\varphi,g)$ *be a three-dimensional contact metric manifold. Then the following statements are equivalent:*

(i) (M,ξ,η,φ,g) *is locally* φ*-symmetric;*

(ii) $\tau = 0$ *and* r *is constant or* $\tau \neq 0$ *and* ξ *belongs to the* (k,μ)*-nullity distribution.*

Proof: For $M = \mathcal{U}_2$ the result is known and has already been mentioned before. Next, we consider the case that $\mathcal{U}_1$ is non-empty.

First, suppose that the contact metric manifold is locally φ-symmetric and consider the orthonormal basis $e_1 = \xi$, $e_2 = e$, $e_3 = \varphi e$. Put $X = \sum X_i e_i$, $Y = \sum Y_i e_i$. Using (2.9), (2.16), we then obtain

$$\begin{aligned} R(X,Y)\xi =& \eta(X)\{(\rho_{11} + \rho_{22} - \frac{r}{2})Y_2 e_2 + (\rho_{11} + \rho_{33} - \frac{r}{2})Y_3 e_3\} \\ &- \eta(Y)\{(\rho_{11} + \rho_{22} - \frac{r}{2})X_2 e_2 + (\rho_{11} + \rho_{33} - \frac{r}{2})X_3 e_3\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &(1-\lambda^2)\{\eta(X)Y - \eta(Y)X\} + 2a\{\eta(X)hY - \eta(Y)hX\} \\ &= \eta(X)\{(1-\lambda^2)Y + 2ahY\} - \eta(Y)\{(1-\lambda^2)X + 2ahX\} \\ &= \eta(X)\{(\rho_{11} + \rho_{22} - \frac{r}{2})Y_2 e_2 + (\rho_{11} + \rho_{33} - \frac{r}{2})Y_3 e_3\} \\ &\quad - \eta(Y)\{(\rho_{11} + \rho_{22} - \frac{r}{2})X_2 e_2 + (\rho_{11} + \rho_{33} - \frac{r}{2})X_3 e_3\}. \end{aligned}$$

Hence, ξ belongs to the (k,μ)-nullity distribution for $k = 1 - \lambda^2$ and $\mu = 2a$. (k and μ are constants because of Theorem 5.1.)

Conversely, let $\xi \in (k,\mu)$-nullity distribution. Then $R(X,Y)\xi = 0$ for $X, Y \in \ker\eta$. In particular, we have

$$\sigma(e) = \rho(e,\xi) = g(R(e,\varphi e)\xi, \varphi e) = 0,$$
$$\sigma(\varphi e) = \rho(\varphi e,\xi) = g(R(\varphi e, e)\xi, e) = 0,$$

and so $\sigma = 0$. Moreover,

$$2(1-\lambda^2) = \rho(\xi,\xi) = R_{1212} + R_{1313} = 2k$$

from which it follows that λ is constant on $\mathcal{U}_1$, and hence $M = \mathcal{U}_1$. Furthermore, using these results and (2.15), a direct computation of $R(X,Y)\xi$ yields $\mu = 2a$ and it follows that a is also constant. As already mentioned at several places, this implies that the manifold is locally homogeneous and then the required result follows by using Theorem 5.1. ∎

Using Theorem 3.5, Theorem 5.1, Theorem 5.2, and $k = 1 - \lambda^2$, $\mu = 2a$, we obtain

THEOREM 5.3: *A non-Sasakian three-dimensional contact metric manifold such that its characteristic vector field belongs to the (k,μ)-nullity distribution is locally isometric to a naturally reductive space if and only if $\mu(\mu+4) = -4k$.*

We refer to [10] for further results about φ-symmetry in contact metric geometry. Furthermore, in [9], it is proved that for general dimension, any non-Sasakian contact metric manifold such that its characteristic vector field belongs to the (k,μ)-nullity distribution is locally homogeneous and also locally φ-symmetric.

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